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ON THE GENERAL CONTACT PROBLEM OF AN INFLATED NONLINEAR PLANE MEMBRANE

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Abstract—The nonaxisymmetric contact problem between an inflated membrane and a rigid indentor is considered. The membrane is assumed to be an initially thin plane sheet. The shape and the boundary of the contact region and the configuration of the deformed membrane under both inflation and indentation are found by employing the minimum potential energy principle subjected to an inequality constraint condition. A slack variable that converts the inequality constraint to an equality constraint condition is introduced. The coordinate functions that describe the deformed configurations of the membrane are assumed to be represented by a series of geometric admissible functions with unknown coefficients. The unknown coefficients that minimize the total potential energy are determined by Fletcher and Powell's[1] iterative descent method for finding the minimum of a function of multivariables.

INTRODUCTION

Several axisymmetric contact problems have been solved recently. The contact problem between an inflated spherical membrane and two parallel rigid plates was solved by Feng and Yang[2]. The indentation of a rigid sphere to a plane circular membrane was solved by Yang and Hsu[3]. In all of these problems, the contact boundaries are determined by a 1-dimensional grid search technique. Since they are axisymmetric problems, the shape of the contact region is determined automatically as soon as the boundary is determined.

The inflation of a neo-Hookean square membrane was solved by Yang and Lu[4]. They obtained the deformed configurations of the inflated membrane by solving a set of partial differential equations. Feng and Huang[5] solved the same problem for a square membrane of Mooney material by employing the minimum potential energy principle. The energy solutions agree with the solutions obtained by equilibrium equations.

The 2-dimensional grid search techniques for nonaxisymmetric contact problems become impractical as far as the computing time is concerned. In this paper, the nonaxisymmetric contact problem between an inflated plane membrane and a rigid indentor is considered. The shape of the contact region and the boundary between the contact and noncontact regions are determined without employing grid search techniques. The configurations for the deformed membrane under both inflation and indentation are determined by the minimum potential energy principle. In the formulation, it is assumed that: the rigid indentor is fixed in space and the inflating membrane is brought into contact with the indentor; there is no friction between contact surfaces; the thickness of the membrane is small compared with other dimensions of the membrane; therefore, the strain variation across the midsurface of the deformed membrane is negligible and the physical quantities across the midsurface of the deformed membrane are equal to the physical quantities at the midsurface of the deformed membrane. The total strain energy for the deformed membrane and the work done by the inflating pressure are calculated. They are functions of the

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deformed coordinates that describe the configurations of the deformed membrane. Due to the presence of the indentor, the membrane may obtain an equilibrium configuration only in the region outside the indentor. Therefore, the equilibrium configuration is subjected to an inequality constraint condition. A slack variable is introduced to convert the inequality constraint condition into an equality constraint condition. The total potential energy is written in terms of the deformed coordinates. The unknown geometric admissible functions for the deformed coordinate and the slack variable are further written in terms of a series of trigonometric functions with unknown coefficients. The unknown coefficients are determined by the minimum potential energy principle. Fletcher and Powell's[1] iterative descent method for finding the minimum for a function of multivariables is employed. The shape of the contact region and the configurations of the deformed membrane are obtained thereafter.

A square plane membrane is considered as an example. The strain–energy density function of the membrane is assumed to have the Mooney form. The rigid indentor is an elliptic paraboloid whose axis of symmetry coincides with the axis perpendicular to the undeformed membrane and passes through the center of the undeformed square membrane.

DEFORMATION ANALYSIS

A point $\bar{p}_0$ in an undeformed plane membrane is located at (x_1, x_2, x_3) as shown in Fig. 1. During deformation, point $\bar{p}_0$ is deformed to a point $\bar{q}_0$ on the deformed membrane. Point $\bar{q}_0$ is located at $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$. $\bar{y}_i$ and x_i are referred to the same coordinate system. The deformation is assumed to be continuous and the mapping between $\bar{p}_0$ and $\bar{q}_0$ is in one-to-one correspondence; therefore, $\bar{y}_i$ are functions of x_i, i.e.

$$d\bar{y}_i = \bar{y}_{i,j}\, dx_j. \tag{1}$$

Indicial notations are used in this paper unless otherwise specified. The Latin indices denote a range of 1, 2, and 3. The Greek indices denote a range of 1 and 2. The repeated indices denote summation. The $(\ )_{,j}$ denotes the partial differentiation with respect to x_j. Another point $\bar{p}_1$ is located in the neighborhood of the point $\bar{p}_0$. $\bar{p}_1$ is located at $(x_1+dx_1, x_2+dx_2, x_3+dx_3)$ in the undeformed membrane. The corresponding point of $\bar{p}_1$ in the deformed configuration is $\bar{q}_1$. $\bar{q}_1$ is located at $(\bar{y}_1+d\bar{y}_1, \bar{y}_2+d\bar{y}_2, \bar{y}_3+d\bar{y}_3)$. The Green strain tensor E_{ij} in terms of $\bar{y}_1$, $\bar{y}_2$ is

$$E_{ij} = \frac{1}{2}(\bar{y}_{k,i}\bar{y}_{k,j} - \delta_{ij}) \tag{2}$$

where δ_{ij} is the Kronecker delta.

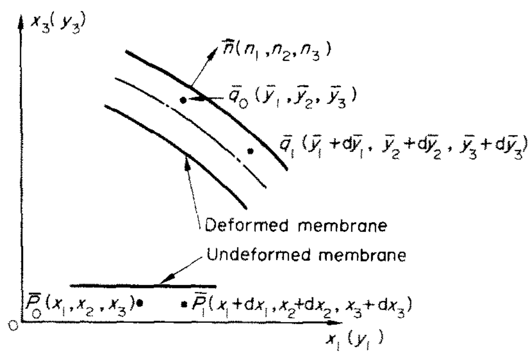

Fig. 1. The geometry of the deformation of a membrane.

The membrane considered in this paper is assumed to be a plane thin membrane. Let the plane $(x_1, x_2, 0)$ coincide with the midsurface of the undeformed membrane as shown in Fig. 2. A point on the undeformed midsurface is described by $(x_1, x_2, 0)$. The corresponding point on the deformed midsurface is described by (y_1, y_2, y_3). A point not on the undeformed midsurface is described by (x_1, x_2, x_3). The corresponding point on the deformed membrane is $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$. Since we assume the thickness of the membrane is small compared with other dimensions of the membrane, physical quantities' variations across the deformed midsurface are neglected and the physical quantities are equal to those on the deformed midsurface. Hence, a segment perpendicular to the undeformed midsurface remains perpendicular to the deformed midsurface during deformation. Based on this relation, we have

$$\bar{y}_i = y_i(x_1, x_2) + \lambda(x_1, x_2)x_3 n_i(x_1, x_2) \tag{3}$$

where λ is the principle stretch ratio in the direction normal to the deformed midsurface. n_i are the components of a unit normal vector $\mathbf{n}$ of the deformed midsurface at point (y_1, y_2, y_3).

Let $\mathbf{i}_i$ be the unit vectors in the x_i direction. Two infinitesimal perpendicular vectors $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_0\mathbf{p}_2$ on the undeformed midsurface pass through point $p_0(x_1, x_2)$ as shown in Fig. 2. For simplicity, let $\mathbf{p}_0\mathbf{p}_1 = \mathrm{d}x_1\mathbf{i}_1$ and $\mathbf{p}_0\mathbf{p}_2 = \mathrm{d}x_2\mathbf{i}_2$. These vectors are stretched and rotated during the deformation. The two new vectors passing through point Q_0 and corresponding to $\mathbf{p}_0\mathbf{p}_1$ and $\mathbf{p}_0\mathbf{p}_2$ are $\mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{Q}_0\mathbf{Q}_2$. Points Q_1 and Q_2 are located at $(y_1 + y_{1,1}\,\mathrm{d}x_1,\ y_2 + y_{2,1}\,\mathrm{d}x_1,\ y_3 + y_{3,1}\,\mathrm{d}x_1)$ and $(y_1 + y_{1,2}\,\mathrm{d}x_2,\ y_2 + y_{2,2}\,\mathrm{d}x_2,\ y_3 + y_{3,2}\,\mathrm{d}x_2)$ respectively. $\mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{Q}_0\mathbf{Q}_2$ are

$$\begin{aligned}\mathbf{Q}_0\mathbf{Q}_1 &= y_{i,1}\,\mathrm{d}x_1\mathbf{i}_i\\ \mathbf{Q}_0\mathbf{Q}_2 &= y_{i,2}\,\mathrm{d}x_2\mathbf{i}_i.\end{aligned} \tag{4}$$

The unit normal vector perpendicular to the midsurface through point Q_0 is

$$\mathbf{n} = \frac{A_i\mathbf{i}_i}{\sqrt{A_jA_j}} \tag{5}$$

where

$$A_i = e_{ijk}y_{j,1}y_{k,2} \tag{6}$$

and e_{ijk} is the permutation symbol.

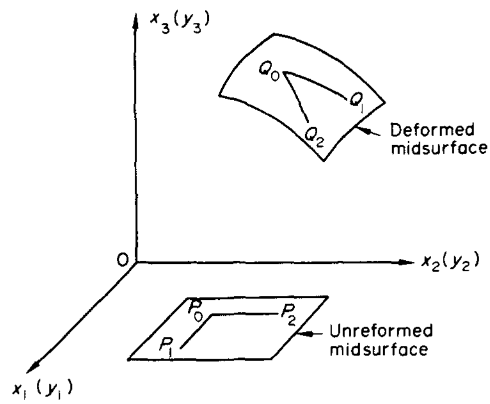

Fig. 2. The geometry of the deformation of the midsurfaces of a membrane.

The incompressibility condition states

$$h_0|\mathbf{p}_0\mathbf{p}_1 \times \mathbf{p}_0\mathbf{p}_2| = \lambda h_0|\mathbf{Q}_0\mathbf{Q}_1 \times \mathbf{Q}_0\mathbf{Q}_2| \tag{7}$$

where h_0 is the thickness of the undeformed membrane at point p_0, hence

$$\lambda = \frac{1}{\sqrt{A_i A_i}}. \tag{8}$$

Equations (5) and (8) relate the unit normal vector of the deformed midsurface and the normal principle stretch ratio to the unknown functions of x_1 and x_2.

THE POTENTIAL ENERGY

The strain energy density function per unit volume for an incompressible material is

$$u = u(I_1, I_2). \tag{9}$$

I_1 and I_2 are first and second strain invariants, and

$$\begin{aligned} I_1 &= 3 + 2E_{ii} \\ I_2 &= 3 + 4E_{rr} + 2(E_{rr}E_{ss} - E_{rs}E_{rs}). \end{aligned} \tag{10}$$

Since the membrane is relatively thin, x_3 is small value, and equation (3) yields

$$\begin{aligned} \bar{y}_{i,\alpha} &= y_{i,\alpha} \\ \bar{y}_{i,3} &= \lambda n_i. \end{aligned} \tag{11}$$

With the relations given in equations (2) and (11), equations (10) reduce to

$$\begin{aligned} I_1 &= y_{1,1}^2 + y_{1,2}^2 + y_{2,1}^2 + y_{2,2}^2 + y_{3,1}^2 + y_{3,2}^2 + (A_1^2 + A_2^2 + A_3^2)^{-1} \\ I_2 &= (y_{1,1}^2 + y_{1,2}^2 + y_{2,1}^2 + y_{2,2}^2 + y_{3,1}^2 + y_{3,2}^2)(A_1^2 + A_2^2 + A_3^2)^{-1} + (A_1^2 + A_2^2 + A_3^2). \end{aligned} \tag{12}$$

The total strain energy for the deformed membrane is

$$U = \int_v u(I_1, I_2)\,\mathrm{d}v \tag{13}$$

where v is the total material volume of the undeformed membrane.

The work done by the inflating pressure, p, during inflation is pV where V is the total volume between the inflated surface and the $y_1 - y_2$ plane; hence

$$W = p\int y_3\,\mathrm{d}R \tag{14}$$

where $\mathrm{d}R$ is the projection of the area formed by two infinitesimal vectors $\mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{Q}_0\mathbf{Q}_2$ onto the $y_1 - y_2$ plane, i.e.

$$dR = (\mathbf{Q_0Q_1} \times \mathbf{Q_0Q_2}) \cdot \mathbf{i}_3. \tag{15}$$

Equation (14) is reduced to

$$W = p \int_{x_2} \int_{x_1} y_3 A_3 \, dx_1 \, dx_2 \tag{16}$$

hence, the total potential energy of the deformed membrane is

$$\Pi = \int_{x_2} \int_{x_1} [h_0 u(I_1, I_2) - p y_3 A_3] \, dx_1 \, dx_2 \tag{17}$$

where Π is then a function of the unknown quantities y_1, y_2 and y_3.

GEOMETRIC CONSTRAINTS

A schematic diagram for an inflated membrane under contact is shown in Fig. 3. The indentor is assumed to be rigid. The contact surfaces are assumed to be frictionless. In this paper, the indentor is assumed to be fixed in space. The plane membrane is brought into contact with the indentor through the inflating process. Since the indentor is assumed to be fixed, no work is done by the indentor during inflation of the membrane. The shape of the indentor is known and can be written as

$$S(y_1, y_2, y_3) = 0. \tag{18}$$

Since the space for the deformed membrane is no longer a free space, the membrane may obtain the equilibrium state only in a region outside the indentor. Therefore, the equilibrium configuration is subjected to the constraint condition

$$S(y_1, y_2, y_3) \leq 0. \tag{19}$$

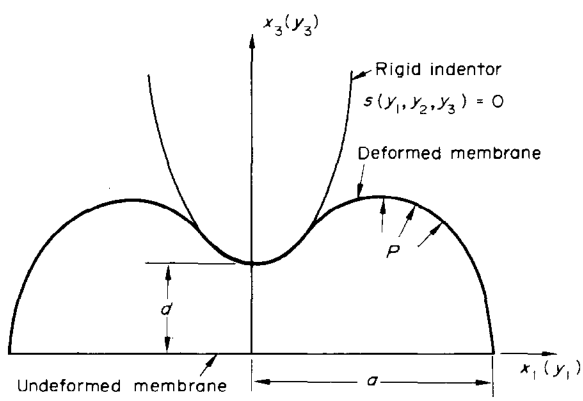

Fig. 3. A schematic diagram of an inflated rectangular membrane under contact.

We introduce a slack variable $\eta(x_1, x_2)$ which converts the inequality constraint condition (19) into the equality constraint condition, i.e.

$$S(y_1, y_2, y_3) + \eta^2 = 0. \tag{20}$$

The slack variable η is a real number for all x_1 and x_2. It is an unknown variable that must be determined.

From the constraint equation (20), y_3 can be solved in terms of y_1, y_2, and

$$y_3 = f(y_1, y_2, \eta). \tag{21}$$

Substituting equation (21) into equation (17), the potential energy reduces to

$$\Pi = \Pi(y_1, y_2, \eta) \tag{22}$$

where y_1, y_2, and η are functions of x_1 and x_2 and are to be determined.

NUMERICAL SOLUTION

The solutions for the functions $y_1(x_1, x_2)$, $y_2(x_1, x_2)$, and $\eta(x_1, x_2)$ are based on the minimum potential energy principle that of all geometrically admissible deformed configurations, the equilibrium configuration minimizes the associated potential energy functional. The geometrically admissible deformed configurations are those satisfying the geometric boundary and geometric constraint conditions. They can be written as a finite series of geometrically admissible functions with unknown coefficients. For determining the unknown functions of y_1, y_2, and η in equation (22), the geometrically admissible functions are

$$\begin{aligned} y_1 &= f_1(x_1, x_2) + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}\phi_{ij}^{(1)}(x_1, x_2) \\ y_2 &= f_2(x_1, x_2) + \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij}\phi_{ij}^{(2)}(x_1, x_2) \\ \eta &= f_3(x_1, x_2) + \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij}\phi_{ij}^{(3)}(x_1, x_2). \end{aligned} \tag{23}$$

Substituting equations (23) into equation (22), the potential energy is reduced to

$$\Pi = \Pi(a_{11}, \ldots, a_{NN}; b_{11}, \ldots, b_{NN}; c_{11}, \ldots, c_{NN}) \tag{24}$$

where the $3 \times N^2$ unknowns a_{ij}, b_{ij}, and $c_{ij} (i = 1, \ldots, N; j = 1, \ldots, N)$ are to be determined. In this paper, these unknowns are determined by Fletcher and Powell's iterative descent method for finding the minimum value of Π. In order to apply Fletcher and Powell's method for determining these unknowns the gradient vector $\mathbf{g}$ of the potential energy functional Π is needed

$$\mathbf{g}^T = \left[\frac{\partial \Pi}{\partial a_{11}}, \ldots, \frac{\partial \Pi}{\partial a_{NN}}; \frac{\partial \Pi}{\partial b_{11}}, \ldots, \frac{\partial \Pi}{\partial b_{NN}}; \frac{\partial \Pi}{\partial c_{11}}, \ldots, \frac{\partial \Pi}{\partial c_{NN}}\right] \tag{25}$$

where $\mathbf{g}^T$ denotes the transpose of a vector $\mathbf{g}$. After evaluating the necessary derivatives for equations (24) and (25), the unknown coefficients for the geometrically admissible functions may be determined by the standard method for minimizing a functional. The IBM SSP FMFP is used for numerical calculations in this paper.

EXAMPLE

A rectangular thin plane membrane with dimensions $2a \times 2b \times h_0$ is clamped to a rectangular hole of the same size. The membrane is inflated by a pressure p and the inflated membrane is brought into contact with an elliptic paraboloid indentor whose axis of symmetry coincides with the axis perpendicular to the undeformed membrane and passes through the center of the undeformed rectangular membrane as shown in Fig. 3. The boundary of the elliptic paraboloid indentor is described by the following equation:

$$(y_3 - d) - a\left(\frac{y_1^2}{r_1^2} + \frac{y_2^2}{r_2^2}\right) = 0 \tag{26}$$

where d is the distance between the tip of the elliptic paraboloid indentor and the center of the undeformed rectangular membrane. r_1 and r_2 are focal lengths in the y_1 and y_2 directions respectively. In the special case $r_1 = r_2$ the indentor is a paraboloid of revolution.

The following dimensionless quantities are used in this example:

$$Y_i = \frac{y_i}{a} \qquad D = \frac{d}{a} \qquad R_\alpha = \frac{r_\alpha}{a}$$
$$L = \frac{b}{a} \qquad P = \frac{pa}{c_1 h_0} \qquad X_\alpha = \frac{x_\alpha}{a}. \tag{27}$$

Due the the presence of the elliptic paraboloid, the constraint equation for the deformed membrane is

$$Y_3 - D - \left(\frac{Y_1^2}{R_1^2} + \frac{Y_2^2}{R_2^2}\right) \le 0. \tag{28}$$

Introducing the slack variable $\eta(X_1, X_2)$ which converts the inequality into the equality condition

$$Y_3 - D - \left(\frac{Y_1^2}{R_1^2} + \frac{Y_1^2}{R_2^2}\right) + \eta^2 = 0 \tag{29}$$

Y_3 can be solved in terms of Y_1, Y_2, and η

$$Y_3 = D + \left(\frac{Y_1^2}{R_1^2} + \frac{Y_2^2}{R_2^2}\right) - \eta^2. \tag{30}$$

For the numerical calculation the strain-energy density function per unit volume of the undeformed membrane is assumed to have the Mooney form

$$u = c_1[I_1 - 3 + \Gamma(I_2 - 3)] \tag{31}$$

where c_1 and Γ are material constants. c_1 is in the units of a stress and Γ is a dimensionless constant.

The total potential energy of the deformed membrane under both inflation and contact with the indentor is

$$\bar{\Pi} = \int_0^L \int_0^1 \left[U(X_1, X_2) - PA_3\left(D + \frac{Y_1^2}{R_1^2} + \frac{Y_2^2}{R_2^2} - \eta^2\right)\right] \mathrm{d}X_1\, \mathrm{d}X_2 \tag{32}$$

where $U(x_1, x_2)$ and $\bar{\Pi}$ are additional two dimensional quantities.

$$U = \frac{u}{c_1} \qquad \bar{\Pi} = \frac{\Pi}{4c_1 h_0 a^2}. \tag{33}$$

The boundary conditions for Y_1, Y_2, and η are

$$\begin{aligned}
X_1 &= \pm 1; \quad & Y_1 &= \pm 1; \quad & Y_2 &= X_2 \\
X_2 &= \pm L; \quad & Y_1 &= X_1; \quad & Y_2 &= \pm L \\
X_1 &= \pm 1; \quad & \eta &= \left[D + \frac{1}{R_1^2} + \frac{X_2^2}{R_2^2}\right]^{1/2} \\
X_2 &= \pm L; \quad & \eta &= \left[D + \frac{X_1}{R_1^2} + \frac{L^2}{R_2^2}\right]^{1/2}.
\end{aligned} \tag{34}$$

Therefore, the geometric admissible functions for Y_1, Y_2, and η are

$$\begin{aligned}
Y_1 &= X_1 + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \sin i\pi X_1 \cos \frac{(2j-1)\pi X_2}{2L} \\
Y_2 &= X_2 + \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} \sin \frac{i\pi X_2}{L} \cos \frac{(2j-1)\pi X_1}{2} \\
\eta &= \left[D + \frac{X_1^2}{R_1^2} + \frac{X_2^2}{R_2^2}\right]^{1/2} - \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} \cos \frac{(2i-1)\pi X_1}{2} \cos \frac{(2j-1)\pi X_2}{2L}.
\end{aligned} \tag{35}$$

Substituting equations (6, 12, 27, 30, 31, 35) and the required derivatives into equation (32), the total potential energy for the deformed membrane is reduced to the form shown in equation (24). With the proper derivatives, the gradient vector g of the potential energy functional (25) can be obtained. The unknown coefficients a_{ij}, b_{ij}, and c_{ij} $(i = 1, \ldots, N;\ j = 1, \ldots, N)$ that minimize the potential energy functional are thereafter calculated by Fletcher and Powell's iterative descent method. Y_3 is then determined by equation (30); hence the equilibrium configurations for a rectangular membrane under both inflations and indentations are obtained.

RESULTS AND DISCUSSION

The results for an inflated square membrane of Mooney materials ($\Gamma = 0{\cdot}1$) under contact are presented in this paper. The square membrane has a dimension $2a \times 2a \times h_0$. The membrane is inflated and brought into contact with an elliptic paraboloid with $R_1 = R_2 = 1$. The boundary between contact and noncontact regions is determined when Y_1, Y_2 and Y_3 satisfy

$$Y_3 - D - (Y_1^2 + Y_2^2) \leq \epsilon \tag{36}$$

where $\epsilon = 0{\cdot}005$ is used for numerical calculations.

The side views for an inflated square membrane under contact for various values of D are shown in Fig. 4. The sequential grid distortion and the constant contour lines for the deformed membrane are shown in Fig. 5. The dotted lines shown on the figures are the boundaries

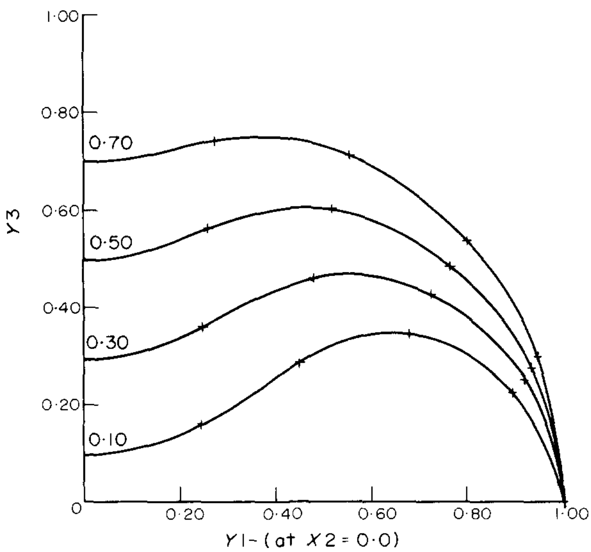

Fig. 4. Profiles of an inflated square Mooney membrane ($\Gamma = 0{\cdot}1$, $p = 2{\cdot}0$) under paraboloid contact.

Table 1. Minimizing coefficients, a_{ij}, for the example ($p = 2{\cdot}0$, $\Gamma = 0{\cdot}1$, $D = 0{\cdot}3$).

j \ i	1		2		3	
1	0·12424	+00	−0·56457	−02	−0·17290	−02
2	−0·40440	−01	0·16829	−02	0·13068	−02
3	−0·11783	−01	0·19286	−02	0·22488	−02

Table 2. Minimizing coefficients, c_{ij}, for the example ($p = 2{\cdot}0$, $\Gamma = 0{\cdot}1$, $D = 0{\cdot}3$).

j \ i	1		2		3	
1	0·43840	+00	0·10708	+00	−0·23836	−01
2	0·10708	+00	0·31997	−01	−0·65264	−02
3	−0·23837	−01	−0·65266	−02	−0·23843	−03

between the contact and noncontact regions. The boundary of contact departs from the circular shape as the contact proceeds. The method presented in this paper is applicable for all values of L, R_1 and R_2. The contact boundary is not a circular shape in general as shown by Feng *et al.*[6].

In this paper the distance D is always positive. For negative values of D, the third equation of (35) must be revised so that the square root is always real. However, the formulation and solution technique remain unchanged.

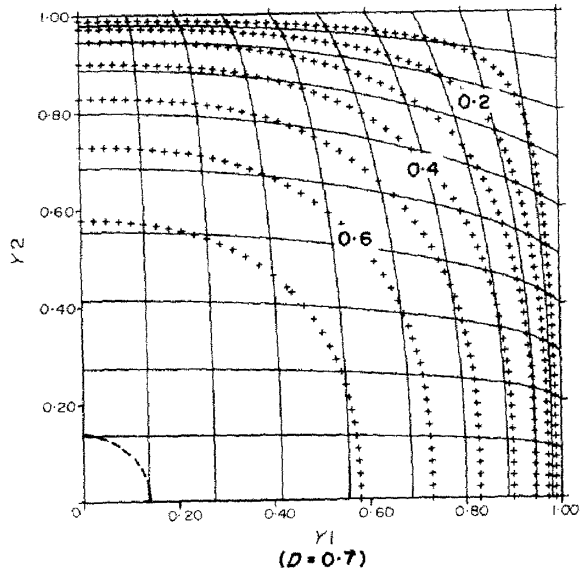

Fig. 5(a).

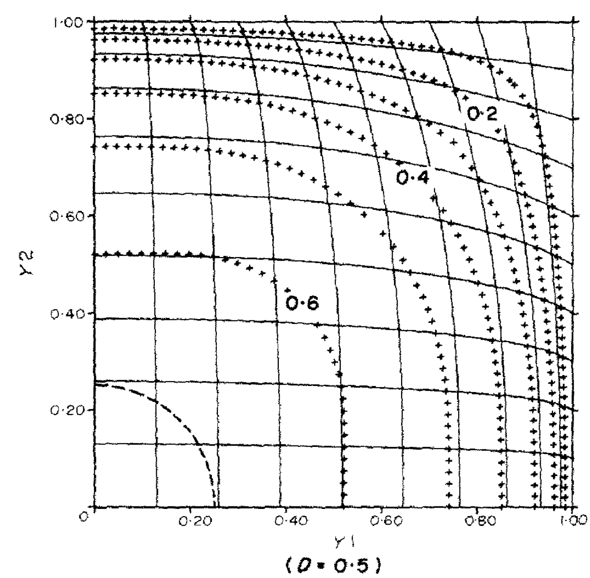

Fig. 5(b).

The error in the solution depends on the number of terms. Tables 1 and 2 list the coefficients a_{ij} and c_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3$). The convergence of the series can be observed. The coefficients b_{ij} are equal to a_{ij} due to symmetry. Also due to symmetry, the coefficient c_{ij} is equal to c_{ji} as shown in the table. The error of the last digit is due to the error in numerical calculation.

The boundary between contact and noncontact regions is sensitive to the values of N.

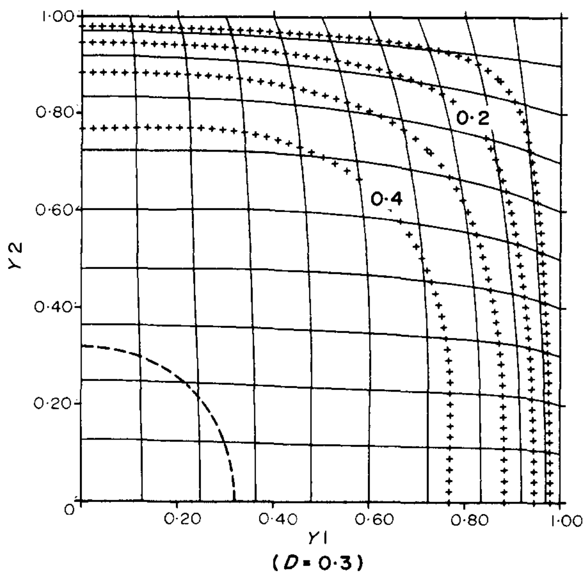

Fig. 5(c).

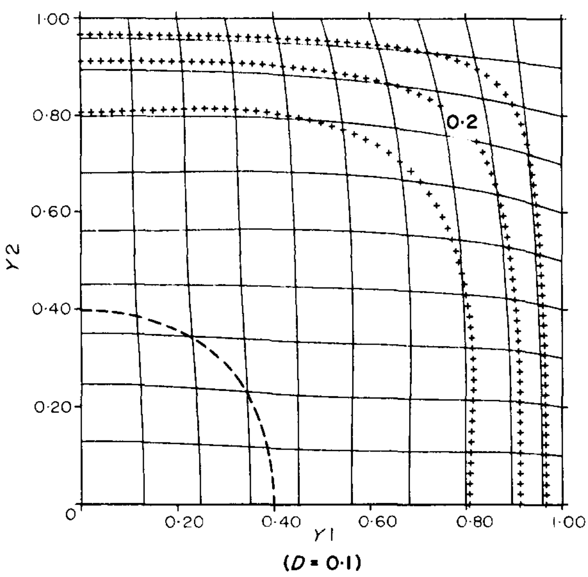

Fig. 5(d).

Fig. 5. Sequantial grid distortions, constant contour lines and contact boundary for a Mooney membrane ($\Gamma = 0{\cdot}1$, $p = 2{\cdot}0$).

However, it converges to the exact solution as the deformation converges to its exact solution. The computing time increases significantly as N increases. For the purpose of illustrating the formulation and the solution technique $N = 3$ is used in this paper.

CONCLUSIONS

The equilibrium equations for membrane problems given by Green and Adkins[7] can be derived from the minimum potential energy principle. The solutions obtained by the energy method and the equilibrium equations should yield the same results. However, it has been demonstrated that the formulation as well as the numerical calculations for the energy method are simpler. This is especially true for nonaxisymmetric contact problems.

Contact problems often arise in studying the mechanics of tires and air bags. This paper offers a method to determine the shapes of the contact region and the configurations of the deformed membranes.

The stress in the membrane can be determined after the deformation is found. The force acting on the indentor is determined thereafter.

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